Tutorial 3 2020/09/30

Outline

- Arc length formula

- Curves by polar coordinates

- Point-set topology on $\mathbb{R}^n$

1) Arc length formula

For $\mathbb{R} \supset I \xrightarrow[\text{of class } C^1]{\vec{x}} \mathbb{R}^n$, $\quad a, b \in I$, $a < b$,

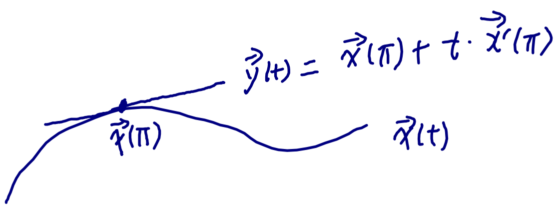
The arclength of $\vec{x}$ for $[a, b]$ is

$$S = \int_a^b \underline{\| x'(t) \|} \, dt$$

Remark : $\vec{x} : I \longrightarrow \mathbb{R}^n$ is of class C^1 if

$\forall \; t \in I$, $\vec{x}'(t)$ exists in $\mathbb{R}^n$ & $\vec{x}'(t) \; I \longrightarrow \mathbb{R}^n$ is continuous.

It guarantee that S is well defined



e.g. $\underline{\vec{x}(t) = (\cos t, \; \sin t, \; t)}$ $\quad t \in [0, 2\pi]$

$\mathbb{R} \longrightarrow \mathbb{R}^3$

- Tangent line of $\vec{x}$ at $t = \pi$

$$\vec{y}(t) = \underset{\text{position}}{\vec{x}(\pi)} + t \; \underset{\text{velocity}}{\vec{x}'(\pi)}$$

$$= (-1, 0, \pi) + t(0, -1, 1)$$

- Arclength

$\underline{\vec{x}'(t) = (-\sin t, \; \cos t, \; 1)}$

check: Arclength of $\vec{x}(t)$ for $[0, 2\pi]$

$= 2\sqrt{2}\,\pi$

Q:

$$\vec{x}(t) : \underset{[s_0, s_1]}{\overset{}{I}} \to \mathbb{R}^n \qquad I = [s_0, s_1]$$

$$\vec{y}(t) : I \to \mathbb{R}^n \qquad I = [t_0, t_1]$$

$$\boxed{s = \int_{s_0}^{s_1} \|\vec{x}'(s)\| \, ds} \qquad \boxed{\tilde{s} = \int_{t_0}^{t_1} \|\vec{y}'(t)\| \, dt} \quad \Rightarrow \quad s = \tilde{s}$$

☆ Arclength is independent of parametrization

pf: Let $s_0 < s_1$, $t_0 < t_1$

$\vec{x}(t)$, $\vec{y}(t)$ be curves of class C^1.

s.t $\exists$ continuously differentiable bijection

$$[s_0, s_1] \xrightarrow{f} [t_0, t_1] \quad w/ \quad f(s_i) = t_i \qquad i = 0, 1$$

s.t $\vec{x} = \vec{y} \circ f$

$$\underbrace{[s_0, s_1] \xrightarrow{f} [t_0, t_1] \xrightarrow{\vec{y}(t)} \mathbb{R}^n}_{\vec{x}(t)}$$

$$S = \int_{s_0}^{s_1} \| \vec{x}'(s) \| \, ds$$

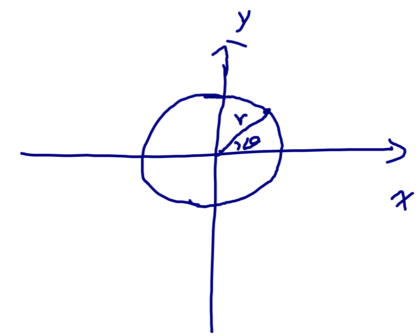
$$= \int_{s_0}^{s_1} \| \vec{y}'(f(\vec{s})) \cdot f'(s) \| \, ds \qquad \text{(chain rule)}$$

$$= \int_{s_0}^{s_1} \| \vec{y}'(f(s)) \| \, |f'(s)| \, ds$$

$$= \int_{s_0}^{s_1} \| \vec{y}'(f(s)) \| \cdot \underbrace{f'(s)\, ds}_{dt}$$

$$= \int_{t_0}^{t_1} \| \vec{y}(t) \| \, dt \qquad \text{by substitution.}$$

e.g. let $r > 0$ Find the circumference of the circle $\boxed{x^2 + y^2 = r^2}$



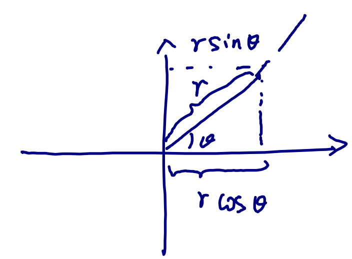
1° parametrize the circle :

write down a simple curve $\vec{x} : [a, b] \to \mathbb{R}^2$ of class C^1

s.t. the image of $\vec{x}$ is the circle.

(r, θ)

Say $\vec{x}(\theta) = (r\cos\theta, r\sin\theta)$, $\theta \in [0, 2\pi]$.



2° Evalue the arclength of $\vec{x}$ for $[0, 2\pi]$

$$\int_0^{2\pi} \| \vec{x}'(\theta) \| \, d\theta$$

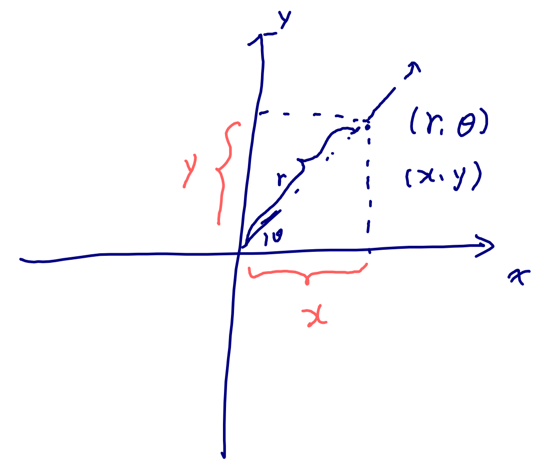
$$= \int_0^{2\pi} \sqrt{(-r\sin\theta)^2 + (r\cos\theta)^2} \, d\theta$$

$$= \int_0^{2\pi} r \, d\theta$$

$$= 2\pi r$$

2) Curves by Polar coordinates :

Two coord in $\mathbb{R}^2$ commoly used :



Rectangular

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$\longleftrightarrow$

Polar

$$\begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \end{cases}$$

$\tan\theta = \frac{y}{x}$

Ex (challenge) :

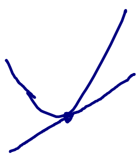
Suppose we have curve in $\mathbb{R}^2$ in polar coord of the form

$(r,\theta) = (f(\theta), \theta)$, $a \le \theta \le b$, f of class C^1.

Show Arclength $= \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

3) Point set topology in $\mathbb{R}^n$

Q What is it & why do we talk about it ?



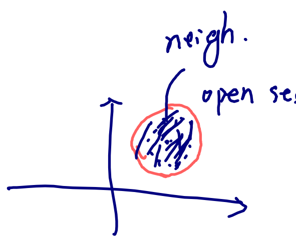
· ·Calculus is an art of approximation

Differentiation $\longleftrightarrow$ curved shapes by tangents

Q: how to discribe two points closely ?

· Suppo we have a set S (say $\mathbb{R}^n$)

Topology concerns a certain collection of subsets of S know as open sets, prescribed by a few axioms.



· Open subsets tell us what is "nearby" a point, & help us to develop the concept of <u>limit</u>.

Excise

Let $S = \{ (\frac{1}{n}, 0) \in \mathbb{R}^2 : n$ is a positive integer $\}$

1) Find $\mathrm{Int}(S)$, $\mathrm{Ext}(S)$ & ∂S

2) Is S open ? is S closed ?

3) Is S connected or disconnected ?

4) Is S bounded or unbounded ?